Linear transformations that are tridiagonal with respect to both eigenbases of a Leonard pair

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Abstract

Let $\mathbb{K}$ denote a field, and let V denote a vector space over $\mathbb{K}$ with finite positive dimension. We consider a pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy (i) and (ii) below:

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.

(ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

We call such a pair a *Leonard pair* on V. Let $\mathcal{X}$ denote the set of linear transformations $X : V \to V$ such that the matrix representing X with respect to the basis (i) is tridiagonal and the matrix representing X with respect to the basis (ii) is tridiagonal. We show that $\mathcal{X}$ is spanned by

$$I,\ A,\ A^*,\ AA^*,\ A^*A,$$

and these elements form a basis for $\mathcal{X}$ provided the dimension of V is at least 3.

1 Leonard pairs

We begin by recalling the notion of a Leonard pair. We will use the following terms. A square matrix X is said to be *tridiagonal* whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. Assume X is tridiagonal. Then X is said to be *irreducible* whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero. We now define a Leonard pair. For the rest of this paper $\mathbb{K}$ will denote a field.

Definition 1.1 [19] Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard pair* on V we mean an ordered pair (A, A^*), where $A : V \to V$ and $A^* : V \to V$ are linear transformations that satisfy (i) and (ii) below:

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.

(ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

Note 1.2 It is a common notational convention to use A^* to represent the conjugate-transpose of A. We are *not* using this convention. In a Leonard pair (A, A^*) the linear transformations A and A^* are arbitrary subject to (i) and (ii) above.

We refer the reader to [3], [9], [12], [13], [14], [15], [16], [17], [18], [19], [21], [22], [23], [24], [25], [26], [27], [28], [30], [31] for background on Leonard pairs. We especially recommend the survey [28]. See [1], [2], [4], [5], [6], [7], [8], [10], [11], [20], [29] for related topics.

2 Leonard systems

When working with a Leonard pair, it is convenient to consider a closely related object called a *Leonard system*. To prepare for our definition of a Leonard system, we recall a few concepts from linear algebra. Let d denote a nonnegative integer and let $\mathrm{Mat}_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra consisting of all $d+1$ by $d+1$ matrices that have entries in $\mathbb{K}$. We index the rows and columns by $0, 1, \ldots, d$. For the rest of this paper, let $\mathcal{A}$ denote a $\mathbb{K}$-algebra isomorphic to $\mathrm{Mat}_{d+1}(\mathbb{K})$, and let V denote a simple $\mathcal{A}$-module. We remark that V is unique up to isomorphism of $\mathcal{A}$-modules, and that V has dimension $d+1$. Let $v_0, v_1, \ldots, v_d$ denote a basis for V. For $X \in \mathcal{A}$ and $Y \in \mathrm{Mat}_{d+1}(\mathbb{K})$, we say Y *represents* X *with respect to* $v_0, v_1, \ldots, v_d$ whenever $Xv_j = \sum_{i=0}^{d} Y_{ij} v_i$ for $0 \leq j \leq d$. For $A \in \mathcal{A}$ we say A is *multiplicity-free* whenever it has $d+1$ mutually distinct eigenvalues in $\mathbb{K}$. Assume A is multiplicity-free. Let $\theta_0, \theta_1, \ldots, \theta_d$ denote an ordering of the eigenvalues of A, and for $0 \leq i \leq d$ put

$$E_i = \prod_{\substack{0 \leq j \leq d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j}, \tag{1}$$

where I denotes the identity of $\mathcal{A}$. We observe (i) $AE_i = \theta_i E_i$ $(0 \leq i \leq d)$; (ii) $E_i E_j = \delta_{i,j} E_i$ $(0 \leq i, j \leq d)$; (iii) $\sum_{i=0}^{d} E_i = I$; (iv) $A = \sum_{i=0}^{d} \theta_i E_i$. Let $\mathcal{D}$ denote the subalgebra of $\mathcal{A}$ generated by A. Using (i)–(iv) we find the sequence $E_0, E_1, \ldots, E_d$ is a basis for the $\mathbb{K}$-vector space $\mathcal{D}$. We call E_i the *primitive idempotent* of A associated with θ_i. It is helpful to think of these primitive idempotents as follows. Observe

$$V = E_0 V + E_1 V + \cdots + E_d V \qquad \text{(direct sum)}. \tag{2}$$

For $0 \leq i \leq d$, $E_i V$ is the (one dimensional) eigenspace of A in V associated with the eigenvalue θ_i, and E_i acts on V as the projection onto this eigenspace. We remark that the $\mathbb{K}$-vector space $\mathcal{D}$ has basis $\{A^i \,|\, 0 \leq i \leq d\}$ and satisfies $\mathcal{D} = \{X \in \mathcal{A} \,|\, AX = XA\}$.

By a *Leonard pair in* $\mathcal{A}$ we mean an ordered pair of elements taken from $\mathcal{A}$ that act on V as a Leonard pair in the sense of Definition 1.1. We now define a Leonard system.

Definition 2.1 [19] By a *Leonard system* in $\mathcal{A}$ we mean a sequence

$$(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

that satisfies (i)–(v) below.

(i) Each of A, A^* is a multiplicity-free element in $\mathcal{A}$.

(ii) $E_0, E_1, \dots, E_d$ is an ordering of the primitive idempotents of A.

(iii) $E_0^*, E_1^*, \dots, E_d^*$ is an ordering of the primitive idempotents of A^*.

(iv) For $0 \leq i, j \leq d$,

$$E_i A^* E_j = \begin{cases} 0 & \text{if } |i-j| > 1, \\ \neq 0 & \text{if } |i-j| = 1. \end{cases} \tag{3}$$

(v) For $0 \leq i, j \leq d$,

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1, \\ \neq 0 & \text{if } |i-j| = 1. \end{cases} \tag{4}$$

Leonard systems are related to Leonard pairs as follows. Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$. Then (A, A^*) is a Leonard pair in $\mathcal{A}$ [27, Section 3]. Conversely, suppose (A, A^*) is a Leonard pair in $\mathcal{A}$. Then each of A, A^* is multiplicity-free [19, Lemma 1.3]. Moreover there exists an ordering $E_0, E_1, \dots, E_d$ of the primitive idempotents of A, and there exists an ordering $E_0^*, E_1^*, \dots, E_d^*$ of the primitive idempotents of A^*, such that $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ is a Leonard system in $\mathcal{A}$ [27, Lemma 3.3].

3 The space $\mathcal{X}$

In this paper we consider a subspace of $\mathcal{A}$ defined as follows.

Definition 3.1 Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$. Let $\mathcal{X}$ denote the $\mathbb{K}$-subspace of $\mathcal{A}$ consisting of the $X \in \mathcal{A}$ such that both

$$E_i X E_j = 0 \qquad \text{if } |i-j| > 1, \tag{5}$$

$$E_i^* X E_j^* = 0 \qquad \text{if } |i-j| > 1 \tag{6}$$

for $0 \leq i, j \leq d$.

We now state our main result.

Theorem 3.2 *Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$. Then the space $\mathcal{X}$ from Definition 3.1 is spanned by*

$$I, \; A, \; A^*, \; AA^*, \; A^*A. \tag{7}$$

Moreover (7) is a basis for $\mathcal{X}$ provided $d \geq 2$.

The proof of Theorem 3.2 will be given in Section 5.

4 The antiautomorphism †

Associated with a given Leonard system in $\mathcal{A}$, there is certain antiautomorphism of $\mathcal{A}$ denoted by † and defined below. Recall an *antiautomorphism* of $\mathcal{A}$ is an isomorphism of $\mathbb{K}$-vector spaces $\sigma : \mathcal{A} \to \mathcal{A}$ such that $(XY)^\sigma = Y^\sigma X^\sigma$ for all $X, Y \in \mathcal{A}$.

Theorem 4.1 *[27, Theorem 7.1] Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$. Then there exists a unique antiautomorphism † of $\mathcal{A}$ such that $A^\dagger = A$ and $A^{*\dagger} = A^*$. Moreover $X^{\dagger\dagger} = X$ for all $X \in \mathcal{A}$.*

Definition 4.2 Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$. We let $\mathcal{D}$ denote the subalgebra of $\mathcal{A}$ generated by A. We let $\mathcal{D}^*$ denote the subalgebra of $\mathcal{A}$ generated by A^*.

Lemma 4.3 *[28, Lemma 6.3] Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$ and let † denote the corresponding antiautomorphism of $\mathcal{A}$ from Theorem 4.1. Then referring to Definition 4.2, † fixes everything in $\mathcal{D}$ and everything in $\mathcal{D}^*$. In particular*

$$E_i^\dagger = E_i, \qquad E_i^{*\dagger} = E_i^* \qquad (0 \le i \le d). \tag{8}$$

5 A basis for $\mathcal{X}$

In this section we prove Theorem 3.2. We start with a lemma.

Lemma 5.1 *[27, Lemma 11.1] Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$ and let V denote a simple $\mathcal{A}$-module. Then $E_iV = E_iE_0^*V$ and $E_i^*V = E_i^*E_0V$ for $0 \le i \le d$.*

Corollary 5.2 *Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$. Then for $Y \in \mathcal{A}$ the following hold for $0 \le i \le d$.*

(i) $YE_i = 0$ if and only if $YE_iE_0^* = 0$.

(ii) $YE_i^* = 0$ if and only if $YE_i^*E_0 = 0$.

Corollary 5.3 *Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$. Then for $Y \in \mathcal{A}$ the following hold for $0 \le i \le d$.*

(i) $E_iY = 0$ if and only if $E_0^*E_iY = 0$.

(ii) $E_i^*Y = 0$ if and only if $E_0E_i^*Y = 0$.

Proof. Apply † to the equations in Corollary 5.2, and use Lemma 4.3. □

Definition 5.4 Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$. For $0 \leq i \leq d$ we let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with E_i (resp. E_i^*). We note that the scalars $\theta_0, \theta_1, \ldots, \theta_d$ (resp. $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$) are mutually distinct and contained in $\mathbb{K}$.

Proposition 5.5 *Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$ and let $\mathcal{X}$ denote the subspace of $\mathcal{A}$ from Definition 3.1. Then for $X \in \mathcal{X}$ such that $XE_0^* = 0$ and $XAE_0^* = 0$ we have $X = 0$.*

Proof. First assume $d = 0$. Then $E_0^* = I$ and the result follows. For the rest of this proof assume $d \geq 1$. We assume $X \neq 0$ and get a contradiction.

In the equation $I = \sum_{i=0}^d E_i^*$ we multiply each term on the right by AE_0^* and simplify the result using (4) to obtain $AE_0^* = E_0^*AE_0^* + E_1^*AE_0^*$; expanding $XAE_0^* = 0$ using this and $XE_0^* = 0$ we find $XE_1^*AE_0^* = 0$. Let V denote a simple $\mathcal{A}$-module and observe $XE_1^*AE_0^*V = 0$. Note that $E_1^*V = E_1^*AE_0^*V$, since $E_1^*AE_0^*V \subseteq E_1^*V$, $\dim E_1^*V = 1$, and $E_1^*AE_0^*V \neq 0$ in view of (4). By the above comments $XE_1^*V = 0$ so $XE_1^* = 0$. In the equation $I = \sum_{i=0}^d E_i^*$ we multiply each term on the left by E_0^*X and simplify the result using (6) to find $E_0^*X = E_0^*XE_0^* + E_0^*XE_1^*$; now $E_0^*X = 0$ since each of XE_0^*, XE_1^* is zero.

Since $X \neq 0$ there exist integers i, j $(0 \leq i, j \leq d)$ such that $E_iXE_j \neq 0$. Define

$$r = \min\{\min\{i,j\} \mid 0 \leq i, j \leq d,\ E_iXE_j \neq 0\}.$$

First assume $r = d$, so that $E_dXE_d \neq 0$ and each of E_iXE_d, E_dXE_i is zero for $0 \leq i \leq d-1$. In the equation $I = \sum_{i=0}^d E_i$ we multiply each term on the left by E_dX and simplify to get $E_dX = E_dXE_d$. By this and since $XE_0^* = 0$ we find $E_dXE_dE_0^* = 0$. Now $E_dXE_d = 0$ by Corollary 5.2(i), for a contradiction.

Next assume $r \leq d-1$. Note that for $0 \leq i \leq r-1$ we have $E_rXE_i = 0$ and $E_iXE_r = 0$. We now show that each of E_rXE_r and E_rXE_{r+1} is zero. In the equation $I = \sum_{i=0}^d E_i$ we multiply each term on the left by E_rX. We simplify the result using (5) and our above comments to find

$$E_rX = E_rXE_r + E_rXE_{r+1}. \tag{9}$$

In this equation we multiply each term on the right by E_0^* and use $XE_0^* = 0$ to find

$$E_rXE_rE_0^* + E_rXE_{r+1}E_0^* = 0. \tag{10}$$

We multiply each term of (9) on the right by A and use $E_iA = \theta_iE_i$ $(0 \leq i \leq d)$ to find $E_rXA = \theta_rE_rXE_r + \theta_{r+1}E_rXE_{r+1}$. In this equation we multiply each term on the right by E_0^* and use $XAE_0^* = 0$ to find

$$\theta_rE_rXE_rE_0^* + \theta_{r+1}E_rXE_{r+1}E_0^* = 0. \tag{11}$$

Solving the linear system (10) and (11), we find $E_rXE_rE_0^* = 0$ and $E_rXE_{r+1}E_0^* = 0$. By this and Corollary 5.2(i) we find $E_rXE_r = 0$ and $E_rXE_{r+1} = 0$. Next we show $E_{r+1}XE_r = 0$. We mentioned earlier that $E_iXE_r = 0$ for $0 \leq i \leq r-1$. In the equation

$I = \sum_{i=0}^{d} E_i$ we multiply each term on the right by XE_r. We simplify the result using (5) and our above comments to find $XE_r = E_{r+1}XE_r$. In this equation we multiply each term on the left by E_0^* and use $E_0^*X = 0$ to find $E_0^*E_{r+1}XE_r = 0$, so $E_{r+1}XE_r = 0$ in view of Corollary 5.3(i). We have now shown that each of E_rXE_r, E_rXE_{r+1}, $E_{r+1}XE_r$ is zero, contracting the definition of r. We conclude $X = 0$. $\square$

Corollary 5.6 *Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$. Then the space $\mathcal{X}$ from Definition 3.1 has dimension at most* 5.

Proof. We assume $d \geq 2$; otherwise $\dim \mathcal{A} \leq 4$ and the result follows. We define linear maps $\pi_0 : \mathcal{X} \to \mathcal{X}E_0^*$ and $\pi_1 : \mathcal{X} \to \mathcal{X}AE_0^*$ by

$$\pi_0(X) = XE_0^*, \qquad \pi_1(X) = XAE_0^* \qquad\qquad (X \in \mathcal{X}).$$

For $i = 0, 1$ let K_i denote the kernel of π_i. We compute the dimensions of K_0 and K_1. First observe

$$\dim E_i^*\mathcal{A}E_j^* = 1 \qquad (0 \leq i, j \leq d).$$

We have $\mathcal{X}E_0^* = E_0^*\mathcal{X}E_0^* + E_1^*\mathcal{X}E_0^*$ in view of (6); therefore $\dim \mathcal{X}E_0^* \leq 2$ so

$$\dim K_0 \geq \dim \mathcal{X} - 2. \tag{12}$$

Combining (4) and (6) we routinely obtain

$$\mathcal{X}AE_0^* \subseteq E_0^*\mathcal{A}E_0^* + E_1^*\mathcal{A}E_0^* + E_2^*\mathcal{A}E_0^*;$$

therefore $\dim \mathcal{X}AE_0^* \leq 3$ so

$$\dim K_1 \geq \dim \mathcal{X} - 3. \tag{13}$$

The intersection of K_0 and K_1 is zero by Propostion 5.5; therefore

$$\dim K_0 + \dim K_1 \leq \dim \mathcal{X}. \tag{14}$$

Combining (12)–(14) we find $\dim \mathcal{X} \leq 5$ as desired. $\square$

Proof of Theorem 3.2. Comparing (3), (4) and (5), (6) we see that each of the elements (7) is contained in $\mathcal{X}$. We must show they actually span $\mathcal{X}$, and that they are linearly independent provided $d \geq 2$. First assume $d = 0$. Then the assertion is obvious. Next assume $d = 1$. Then one routinely verifies that $\mathcal{X} = \mathcal{A}$ is spanned by the elements (7). Finally assume $d \geq 2$. In view of Corollary 5.6, it suffices to show that the elements (7) are linearly independent. Suppose

$$eI + fA + f^*A^* + gAA^* + g^*A^*A = 0 \tag{15}$$

for some scalars e, f, f^*, g, g^* in $\mathbb{K}$. We show each of e, f, f^*, g, g^* is zero. For $1 \leq i \leq d$ we multiply each term in (15) on the left by E_{i-1}^* and the right by E_i^* to obtain

$$(f + g\theta_i^* + g^*\theta_{i-1}^*)E_{i-1}^*AE_i^* = 0.$$

By this and since $E^*_{i-1}AE^*_i$ is nonzero we find

$$f + g\theta^*_i + g^*\theta^*_{i-1} = 0 \qquad (1 \leq i \leq d). \tag{16}$$

For $1 \leq i \leq d$ we multiply each term in (15) on the left by E^*_i and the right by E^*_{i-1} to obtain

$$(f + g\theta^*_{i-1} + g^*\theta^*_i)E^*_iAE^*_{i-1} = 0.$$

By this and since $E^*_iAE^*_{i-1}$ is nonzero we find

$$f + g\theta^*_{i-1} + g^*\theta^*_i = 0 \qquad (1 \leq i \leq d). \tag{17}$$

Combining (16) at $i = 1$ and (17) at $i = 1, 2$ we routinely find that each of f, g, g^* is zero. Interchanging the roles of A and A^* in the above argument we find $f^* = 0$. Now (15) becomes $eI = 0$ so $e = 0$. We have now shown that each of e, f, f^*, g, g^* is zero and the result follows. $\square$

6 The linear maps Υ and Υ^*

In this section we discuss some linear maps $\Upsilon : \mathcal{X} \to \mathcal{D}$ and $\Upsilon^* : \mathcal{X} \to \mathcal{D}^*$ that we find attractive. To motivate things we recall some results by the second author and Vidunas.

Lemma 6.1 *[30, Theorem 1.5] Let $(A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$. Then there exists a sequence of scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^*$ taken from $\mathbb{K}$ such that both*

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^* = \gamma^*A^2 + \omega A + \eta I, \tag{18}$$

$$A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^*A = \gamma A^{*2} + \omega A^* + \eta^* I. \tag{19}$$

Moreover the sequence is uniquely determined by the Leonard system provided $d \geq 3$.

Note 6.2 The equations (18) and (19) first appeared in [32]; they are called the *Askey-Wilson relations.*

We have a comment.

Lemma 6.3 *[30, Theorem 4.5] Let $(A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$. Then referring to Definition 5.4 and Lemma 6.1 we have*

$$\begin{aligned}
\beta + 1 &= \frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} = \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \qquad (2 \leq i \leq d-1), && (20)\\
\gamma &= \theta_{i-1} - \beta\theta_i + \theta_{i+1} \qquad (1 \leq i \leq d-1), && (21)\\
\gamma^* &= \theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1} \qquad (1 \leq i \leq d-1), && (22)\\
\varrho &= \theta^2_{i-1} - \beta\theta_{i-1}\theta_i + \theta^2_i - \gamma(\theta_{i-1} + \theta_i) \qquad (1 \leq i \leq d), && (23)\\
\varrho^* &= \theta^{*2}_{i-1} - \beta\theta^*_{i-1}\theta^*_i + \theta^{*2}_i - \gamma^*(\theta^*_{i-1} + \theta^*_i) \qquad (1 \leq i \leq d). && (24)
\end{aligned}$$

Theorem 6.4 *Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$. Let the spaces $\mathcal{X}$ and $\mathcal{D}$ be as in Definitions 3.1 and 4.2, respectively. Then there exists a $\mathbb{K}$-linear map $\Upsilon : \mathcal{X} \to \mathcal{D}$ that satisfies*

$$\Upsilon(X) = A^2X - \beta AXA + XA^2 - \gamma(AX + XA) - \varrho X \tag{25}$$

for all $X \in \mathcal{X}$. Moreover

$$\begin{aligned}
\Upsilon(I) &= (2-\beta)A^2 - 2\gamma A - \varrho I, && (26)\\
\Upsilon(A) &= (2-\beta)A^3 - 2\gamma A^2 - \varrho A, && (27)\\
\Upsilon(A^*) &= \gamma^* A^2 + \omega A + \eta I, && (28)\\
\Upsilon(AA^*) &= \gamma^* A^3 + \omega A^2 + \eta A, && (29)\\
\Upsilon(A^*A) &= \gamma^* A^3 + \omega A^2 + \eta A. && (30)
\end{aligned}$$

Proof. Certainly there exists a $\mathbb{K}$-linear map $\Upsilon : \mathcal{X} \to \mathcal{A}$ that satisfies (25). Using (18) we find Υ satisfies (26)–(30). Combining (26)–(30) and Theorem 3.2 we find $\Upsilon(X) \in \mathcal{D}$ for all $X \in \mathcal{X}$, and the result follows. $\Box$

Interchanging the roles of A and A^* in Theorem 6.4 we obtain:

Theorem 6.5 *Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$. Let the spaces $\mathcal{X}$ and $\mathcal{D}^*$ be as in Definitions 3.1 and 4.2, respectively. Then there exists a $\mathbb{K}$-linear map $\Upsilon^* : \mathcal{X} \to \mathcal{D}^*$ that satisfies*

$$\Upsilon^*(X) = A^{*2}X - \beta A^*XA^* + XA^{*2} - \gamma^*(A^*X + XA^*) - \varrho^* X \tag{31}$$

for all $X \in \mathcal{X}$. Moreover

$$\begin{aligned}
\Upsilon^*(I) &= (2-\beta)A^{*2} - 2\gamma^* A^* - \varrho^* I,\\
\Upsilon^*(A^*) &= (2-\beta)A^{*3} - 2\gamma^* A^{*2} - \varrho^* A^*,\\
\Upsilon^*(A) &= \gamma A^{*2} + \omega A^* + \eta^* I,\\
\Upsilon^*(A^*A) &= \gamma A^{*3} + \omega A^{*2} + \eta^* A,\\
\Upsilon^*(AA^*) &= \gamma A^{*3} + \omega A^{*2} + \eta^* A.
\end{aligned}$$

We have a comment concerning the image and kernel of Υ.

Lemma 6.6 *Referring to Theorem 6.4 the following (i)–(iii) hold.*

(i) $\mathrm{Span}\{AA^* - A^*A\} \subseteq \mathrm{Ker}(\Upsilon)$.

(ii) $\mathrm{Im}(\Upsilon) \subseteq \mathrm{Span}\{I, A, A^2, A^3\}$.

(iii) Assume $d \geq 3$. Then equality holds in (i) if and only if equality holds in (ii).

Proof. (i), (ii): Immediate from Theorem 6.4.
(iii): Use Theorem 3.2 and elementary linear algebra. $\Box$

Interchanging the roles of A and A^* in Lemma 6.6 we obtain:

Lemma 6.7 *Referring to Theorem 6.5 the following (i)–(iii) hold.*

(i) $\mathrm{Span}\{AA^* - A^*A\} \subseteq \mathrm{Ker}(\Upsilon^*)$.

(ii) $\mathrm{Im}(\Upsilon^*) \subseteq \mathrm{Span}\{I, A^*, A^{*2}, A^{*3}\}$.

(iii) Assume $d \geq 3$. Then equality holds in (i) if and only if equality holds in (ii).

Referring to Lemmas 6.6 and 6.7 it appears that we have equality in (i) and (ii) for most Leonard systems but not all. Below we give an example where equality is not attained.

Definition 6.8 Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$. We say this Leonard system is *bipartite* (resp. *dual bipartite*) whenever $E_i^* A E_i^* = 0$ (resp. $E_i A^* E_i = 0$) for $0 \leq i \leq d$.

Lemma 6.9 *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system in $\mathcal{A}$. Then referring to Theorems 6.4, 6.5 and Definition 6.8 the following (i), (ii) hold provided $d \geq 3$.*

(i) Assume Φ is bipartite. Then

$$\begin{aligned} \mathrm{Ker}(\Upsilon^*) &= \mathrm{Span}\{A, AA^*, A^*A\}, \\ \mathrm{Im}(\Upsilon^*) &= \mathrm{Span}\{B^*, A^*B^*\}, \end{aligned}$$

where $B^* = (2-\beta)A^{*2} - 2\gamma^* A^* - \varrho^* I$.

(ii) Assume Φ is dual bipartite. Then

$$\begin{aligned} \mathrm{Ker}(\Upsilon) &= \mathrm{Span}\{A^*, A^*A, AA^*\}, \\ \mathrm{Im}(\Upsilon) &= \mathrm{Span}\{B, AB\}, \end{aligned}$$

where $B = (2-\beta)A^2 - 2\gamma A - \varrho I$.

Proof. (ii): By [12] and [30, Theorem 5.3] each of γ^*, ω, η is zero. By this and Theorem 6.4 we have $\mathrm{Ker}(\Upsilon) \supseteq \mathrm{Span}\{A^*, A^*A, AA^*\}$ and $\mathrm{Im}(\Upsilon) = \mathrm{Span}\{B, AB\}$. To show $\mathrm{Ker}(\Upsilon) = \mathrm{Span}\{A^*, A^*A, AA^*\}$ it suffices to show that B and AB are linearly independent. Suppose B and AB are linearly dependent. Then $B = 0$ since the elements I, A, A^2, A^3 are linearly independent. Since $d \geq 3$ there exists an integer i such that $1 \leq i \leq d-1$. Multiplying each term in the equation $B = (2-\beta)A^2 - 2\gamma A - \varrho I$ by E_i and simplifying we find E_i times

$$(2-\beta)\theta_i^2 - 2\gamma\theta_i - \varrho \tag{32}$$

is zero. Of course E_i is not zero so (32) is zero. Using (21) and (23) we routinely find (32) is equal to

$$(\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1})$$

and is therefore nonzero. This is a contradiction and the result follows. $\square$

Open Problem: Referring to Lemmas 6.6 and 6.7, precisely determine the set of Leonard systems for which equality holds in (i) and (ii).

References

[1] H. Alnajjar and B. Curtin. A family of tridiagonal pairs. *Linear Algebra Appl.* **390** (2004) 369–384.

[2] H. Alnajjar and B. Curtin. A family of tridiagonal pairs related to the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$. *Electron. J. Linear Algebra* **13** (2005) 1–9.

[3] B. Hartwig. Three mutually adjacent Leonard pairs. *Linear Algebra Appl.* **408** (2005) 19–39; `arXiv:math.AC/0508415`.

[4] T. Ito, K. Tanabe, and P. Terwilliger. Some algebra related to P- and Q-polynomial association schemes, in: *Codes and Association Schemes (Piscataway NJ, 1999)*, Amer. Math. Soc., Providence RI, 2001, pp. 167–192; `arXiv:math.CO/0406556`.

[5] T. Ito and P. Terwilliger. The shape of a tridiagonal pair. *J. Pure Appl. Algebra* **188** (2004) 145–160; `arXiv:math.QA/0304244`.

[6] T. Ito and P. Terwilliger. Tridiagonal pairs and the quantum affine algebra $U_q(\widehat{sl}_2)$. *Ramanujan J.* In press; `arXiv:math.QA/0310042`.

[7] T. Ito and P. Terwilliger. Two non-nilpotent linear transformations that satisfy the cubic q-Serre relations. Submitted to *J. Algebra Appl.*; `arXiv:math.QA/0508398`.

[8] T. Ito, P. Terwilliger and C. Weng. The quantum algebra $U_q(sl_2)$ and its equitable presentation. *J. Algebra* **298** (2006) 284–301; `arXiv:math.QA/0507477`.

[9] K. Nomura. Tridiagonal pairs and the Askey-Wilson relations. *Linear Algebra Appl.* **397** (2005) 99–106.

[10] K. Nomura. A refinement of the split decomposition of a tridiagonal pair. *Linear Algebra Appl.* **403** (2005) 1–23.

[11] K. Nomura. Tridiagonal pairs of height one. *Linear Algebra Appl.* **403** (2005) 118–142.

[12] K. Nomura and P. Terwilliger. Balanced Leonard pairs. *Linear Algebra Appl.* Submitted; `arXiv:math.RA/0506219`.

[13] K. Nomura and P. Terwilliger. Some trace formulae involving the split sequences of a Leonard pair. *Linear Algebra Appl.* **413** (2006), 189–201; `arXiv:math.RA/0508407`.

[14] K. Nomura and P. Terwilliger. The determinant of $AA^* - A^*A$ for a Leonard pair A, A^*. *Linear Algebra Appl.* In press; `arXiv:math.RA/0511641`.

[15] K. Nomura and P. Terwilliger. Matrix units associated with the split basis of a Leonard pair. *Linear Algebra Appl.* In press; `arXiv:math.RA/0602416`.

[16] A. A. Pascasio. On the multiplicities of the primitive idempotents of a Q-polynomial distance-regular graph. *European J. Combin.* **23** (2002) 1073–1078.

[17] P. Terwilliger. The subconstituent algebra of an association scheme I. *J. Algebraic Combin.* **1** (1992) 363–388.

[18] P. Terwilliger. The subconstituent algebra of an association scheme III. *J. Algebraic Combin.* **2** (1993) 177–210.

[19] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other. *Linear Algebra Appl.* **330** (2001) 149–203; `arXiv:math.RA/0406555`.

[20] P. Terwilliger. Two relations that generalize the q-Serre relations and the Dolan-Grady relations. In *Physics and Combinatorics 1999 (Nagoya)*, 377–398, World Scientific Publishing, River Edge, NJ, 2001; `arXiv:math.QA/0307016`.

[21] P. Terwilliger. Leonard pairs from 24 points of view. *Rocky Mountain J. Math.* **32**(2) (2002) 827–888; `arXiv:math.RA/0406577`.

[22] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other; the TD-D and the LB-UB canonical form. *J. Algebra.* **291** (2005) 1–45; `arXiv:math.RA/0304077`.

[23] P. Terwilliger. Introduction to Leonard pairs. OPSFA Rome 2001. *J. Comput. Appl. Math.* **153**(2) (2003) 463–475.

[24] P. Terwilliger. Introduction to Leonard pairs and Leonard systems. *Sūrikaisekikenkyūsho Kōkyūroku*, (1109) 67–79, 1999. Algebraic Combinatorics (Kyoto, 1999).

[25] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other; comments on the split decomposition. *J. Comput. Appl. Math.* **178** (2005) 437–452; `arXiv:math.RA/0306290`.

[26] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other; comments on the parameter array. *Des. Codes Cryptogr.* **34** (2005) 307–332; `arXiv:math.RA/0306291`.

[27] P. Terwilliger. Leonard pairs and the q-Racah polynomials. *Linear Algebra Appl.* **387** (2004) 235–276; `arXiv:math.QA/0306301`.

[28] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other; an algebraic approach to the Askey scheme of orthogonal polynomials. Lecture notes for the summer school on orthogonal polynomials and special functions. Universidad Carlos III de Madrid, Leganes, Spain. July 8–July 18, 2004; `arXiv:math.QA/0408390`.

[29] P. Terwilliger. The equitable presentation for the quantum group $U_q(g)$ associated with a symmetrizable Kac-Moody algebra g. *J. Algebra* **298** (2006) 302–319; `arXiv:math.QA/0507478`.

[30] P. Terwilliger and R. Vidunas. Leonard pairs and the Askey-Wilson relations. *J. Algebra Appl.* **3** (2004) 411–426; `arXiv:math.QA/0305356`.

[31] R. Vidunas. Normalized Leonard pairs and Askey-Wilson relations. Preprint; `arXiv:math.RA/0505041`.

[32] A.S. Zhedanov. "Hidden symmetry" of Askey-Wilson polynomials. *Teoret. Mat. Fiz.* **89** (1991) 190–204.

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